Dependency of resolvable gravitational spatial resolution on space-borne observation techniques

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Abstract. The so-called Colombo-Nyquist (Colombo (1984)) rule in satellite geodesy has been revisited. This rule predicts that for a gravimetric satellite flying in a (near-)polar circular repeat orbit, the maximum resolvable geopotential spherical harmonic degree ($l_{\max}$) is equal to half the number of orbital revolutions (n_{r}) the satellite completes in one repeat period. This rule has been tested for different observation types, including geoid values at sea level along the satellite ground track, orbit perturbations (radial, along-track, cross-track), low-low satellite-to-satellite tracking, and satellite gravity gradiometry observations (all three diagonal components). Results show that the Colombo-Nyquist must be reformulated. Simulations indicate that the maximum resolvable degree is in fact equal to $kn_{\mathrm{r}} + 1$, where k can be equal to 1, 2, or even 3 depending on the combination of observation types. However, the original rule is correct to some extent, considering that the quality of recovered gravity field models is homogeneous as a function of geographical longitude as long as $l_{\max} < n_{\mathrm{r}}/2$.

Keywords. Colombo-Nyquist rule, maximum resolvable degree, space-borne gravimetry

1 Introduction

Colombo (1984) has indicated that for exact satellite circular repeat orbits and for continuous space-borne gravimetric observations, the normal matrix of gravity field spherical harmonic (SH) coefficients becomes block-diagonal when organized per SH order. The correlation between different orders is zero as long as one can avoid overlapping frequencies, which is generally guaranteed if the maximum resolvable SH degree ($l_{\max}$) is less than half the number of orbital revolutions n_{r} which the satellite completes in a repeat period of n_{d} nodal days, or $l_{\max} < n_{\mathrm{r}}/2$ (Schrama (1990)). Although Sneeuw (2000) has pointed out that avoiding overlapping frequencies is fundamentally a restriction on the maximum SH order. Nevertheless this has led to the rule-of-thumb that the maximum resolvable degree is equal to $n_{\mathrm{r}}/2$, referred to as the Colombo-Nyquist rule. This rule has major implications for the design of future gravity field missions, where several trade-offs have to be made, such as temporal and spatial resolution, the observation/decoupling of different sources of gravity field changes, etc. (Bender et al. (2008); Reubelt et al. (2009); Visser and Schrama (2005)). Also, this rule has implications for designing efficient gravity field estimation schemes taking advantage of the structure of normal matrices (Schrama (1991)). It has to be noted that the maximum resolvable degree is defined as the maximum SH degree for which also all coefficients with SH orders complete to this maximum degree can be resolved. It is thus not precluded that certain individual coefficients with a higher SH degree can be resolved, however with a SH order that is not higher.

The Colombo-Nyquist rule-of-thumb has been tested for a number of mission scenarios, i.e. different repeat orbits and combinations of observables. It is shown that this rule needs to be reformulated. The selected mission scenarios are outlined in Section 2. The method used for establishing the maximum resolvable degree for these mission scenarios is briefly described in Section 3. Results are presented in Section 4 and summarized in Section 5.

2 Mission scenarios

The selected repeat orbits and observable types are listed in Table 1. The orbits are polar to ensure global coverage. A repeat orbit is specified by the number of revolutions n_{r} that is completed in n_{d} nodal days, where n_{r} and n_{d} do not have common prime factors (except 1). Short repeat periods ranging from 1 to 3 days have been selected to limit the computational burden. These short repeat periods are however sufficient to test the validity of the Colombo-Nyquist rule. Different parities for n_{r} and n_{d} were selected to assess the possible impact on the maximum resolvable degree of the number of distinct equator crossings.

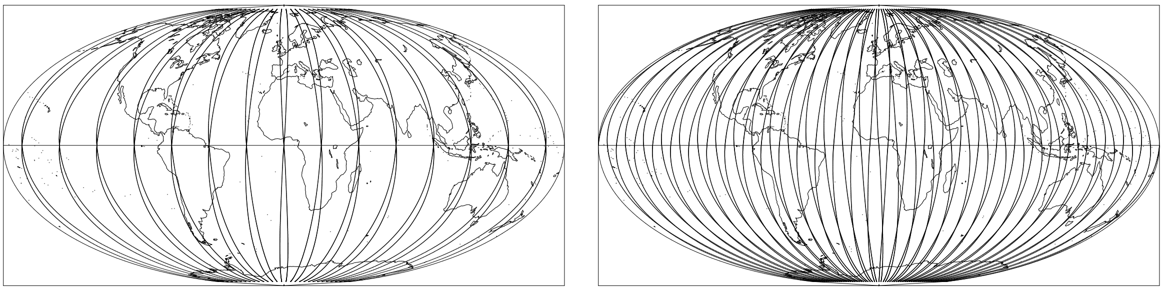

Fig. 1. Ground track pattern for 15/1 (left) and 31/2 (right) polar repeat orbits.

Table 1. Selected polar repeat orbits and observation techniques. The time interval between observations is always taken equal to 1 s.

Repeat period n_d (days)	Number of revolutions n_r	Height (km)
1	15	554.25
2	31	404.35
3	46	453.41
3	47	356.16

Observation technique	Precision level
Geoid	1 cm
Orbit	1 cm
ll-SST	1 μm
SGG	0.01 E

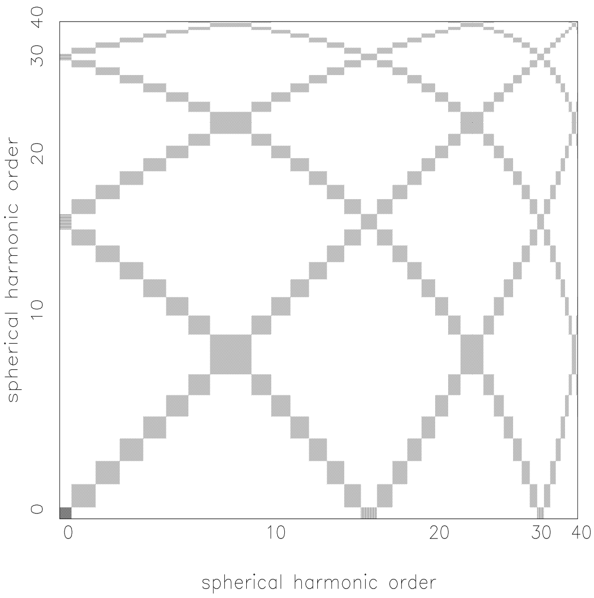

Fig. 2. Structure of normal matrix for gravity field coefficients complete to degree and order 40 for a 15/1 polar repeat orbit based on ll-SST observations (”kite matrix”).

For $n_\mathrm{r} - n_\mathrm{d}$ even the number of equator crossings is equal to n_r, whereas this is $2n_\mathrm{r}$ for $n_\mathrm{r} - n_\mathrm{d}$ odd (Fig. 1).

The observable types include geoid values at sea level along the satellite ground path (closely related to altimeter observations), orbit perturbations in the radial, along-track and cross-track direction, low-low satellite-to-satellite tracking (ll-SST) range observations, and satellite gravity gradient (SGG) observations (the diagonal components, where the gradiometer instrument is aligned with the radial, along-track, and cross-track direction). The observations are assumed to be provided continuously with a constant time step of 1 s. The relation between SH gravity field coefficients and observations is given by well-established and tested transfer functions (e.g. Schrama (1991); Sneeuw (2000); Visser (1992); Visser et al. (1994, 2001, 2003); Visser (2005)). These transfer functions are used to set up the observation equations, which are to be solved by the weighted least-squares method (Section 3). The observations are assigned weights in accordance with the precision levels listed in Table 1.

3 Estimating the maximum resolvable spherical harmonic degree

For a repeat orbit, Colombo (1984) indicated that when a least-squares estimation method is used and if a continuous time series of observations is obtained with constant time interval, the normal matrix for the SH coefficients will become block-diagonal when organized per order, and correlations between different orders will be equal to zero as long as the maximum resolvable degree is below $n_\mathrm{r}/2$. For higher degrees, different orders get correlated and the normal matrix adopts a Kite-like structure (e.g. Fig. 2). The question is addressed if still a stable gravity field solution can be obtained in the presence of these correlations, thereby assuming that no use is made of prior knowledge and/or regularization. This is tested by computing the condition number of this matrix (ratio of maximum and minimum eigenvalue) and by comput-

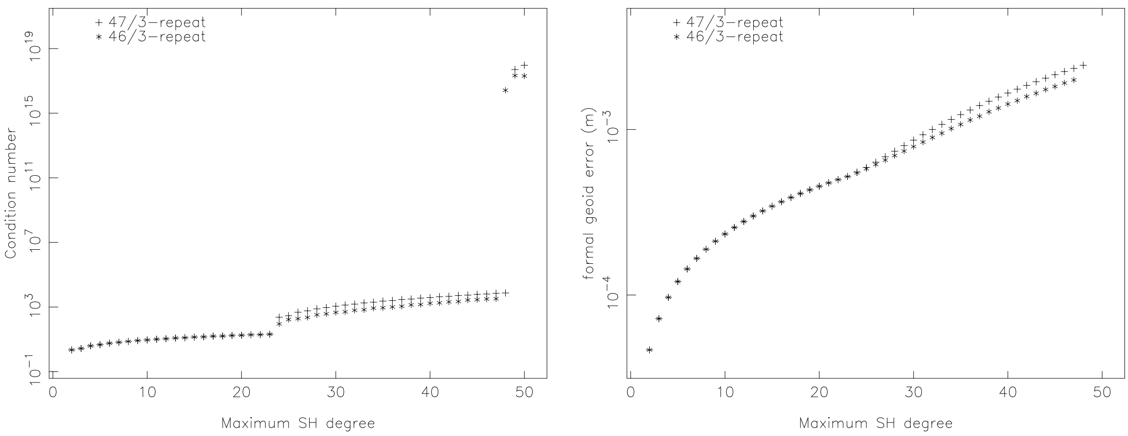

Fig. 3. Condition number of the normal equations (left) and global RMS formal geoid error as a function of the maximum retrieved spherical harmonic degree. Use is made of geoid observations at sea level.

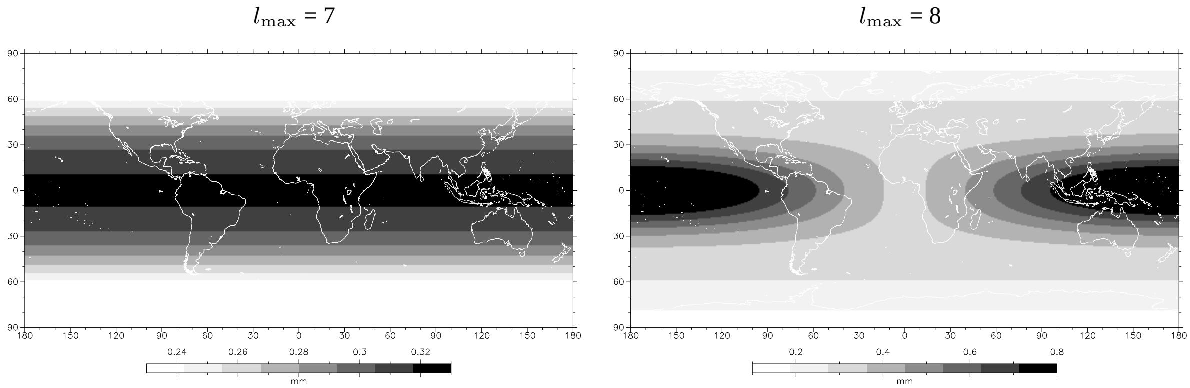

Fig. 4. Formal geoid error for 15/1-repeat for geoid observations as a function of the geographical location.

ing the Root-Mean-Square (RMS) of the cumulative global formal geoid commission error for the estimated SH coefficients. The formal geoid errors were taken from the inverse (if the normal matrix is invertible) of the weighted normal matrix. In all cases, normal equations were set up for all SH coefficients from degree 2 to a certain maximum degree $l_{\max}$. Thus the impact of omission and/or aliasing of unmodeled gravity field sources are not taken into account. The exercises described in this paper only address the issue of observability of a static gravity field complete to the maximum SH degree solved for.

4 Gravity field observability

As a first test case, the condition numbers of the normal matrix and associated geoid error were computed for 46/3 and 47/3 repeat orbits using geoid observations along the ground track. The condition numbers display a large jump at $l_{\max} = n_{\mathrm{r}}$ (Fig. 3, left) and in fact the normal matrix could not be inverted for higher degrees (no formal geoid errors could be estimated, Fig. 3, right). For $l_{\max} = n_{\mathrm{r}}/2$, a small jump in the condition number occurs due to the additional correlations between different SH orders, but this does not lead to an unstable normal matrix. Also, the slope of the geoid error increases for $l_{\max} > n_{\mathrm{r}}/2$. Based on these results, it can already be concluded that the maximum resolvable degree can be as big as n_{r} and does not depend on the parity of n_{r} and n_{d}.

It is interesting to note that as long as $l_{\max} < n_{\mathrm{r}}/2$, the geoid error is only latitude dependent and does not change with longitude, whereas for $l_{\max} > n_{\mathrm{r}}/2$ the correlations between different orders cause the geoid error to change as a function of longitude as well (Fig. 4). However, this rule does not have a universal validity as well and depends on the parity of n_{r} and n_{d}. In fact, this rule applies only for $n_{\mathrm{r}} - n_{\mathrm{d}}$ even. For $n_{\mathrm{r}} - n_{\mathrm{d}}$ odd, the geoid error does not depend on the longitude for $l_{\max} < n_{\mathrm{r}}$ (Table 2). The variation of the geoid error as a function of latitude and lon-

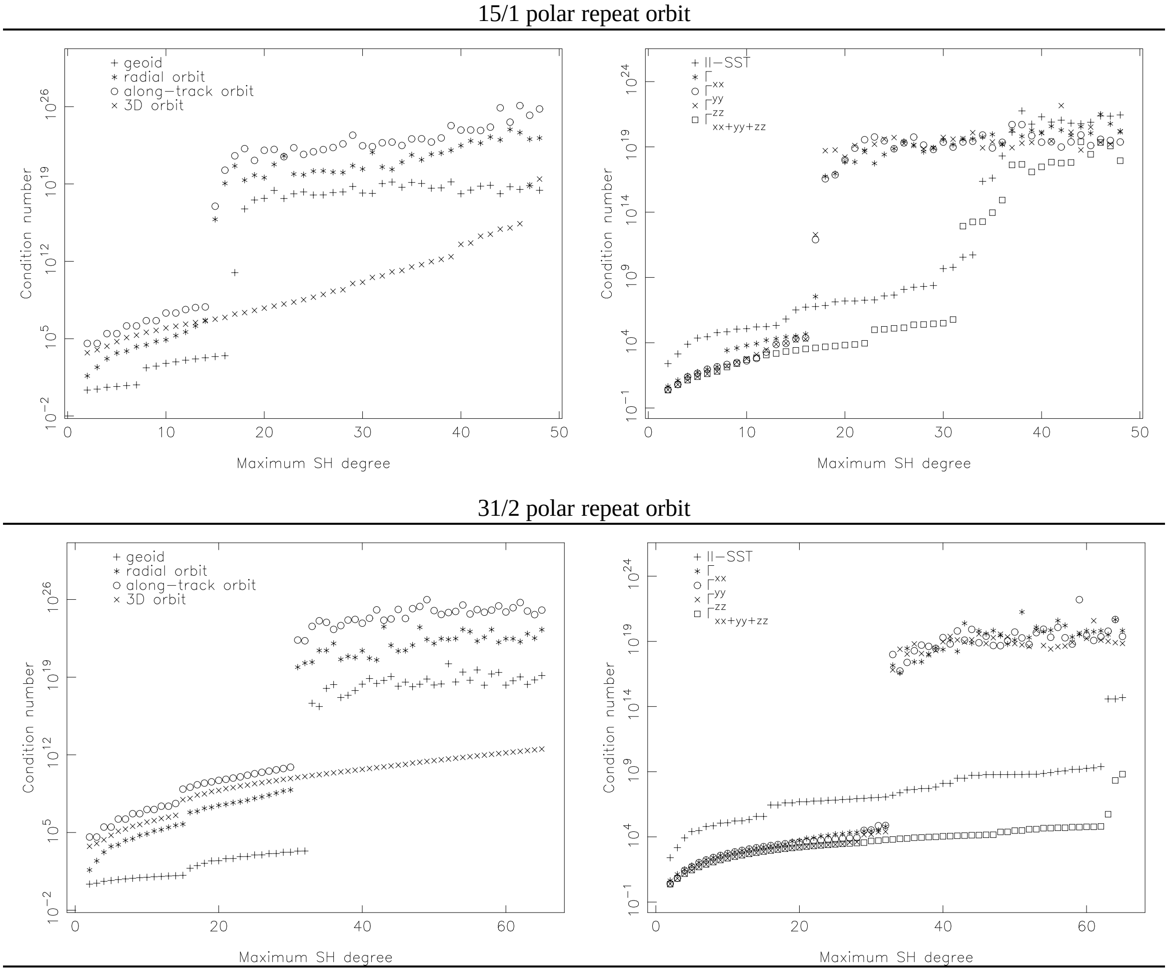

Fig. 5. Condition number of the normal equations as a function of the maximum retrieved spherical harmonic degree and the observation technique (the minimum degree is equal to 2).

gitude depends on the observable. For a 15/1-repeat orbit and geoid observations, the minimum and maximum formal geoid error is equal to 0.19 and 6.42 mm for a gravity field recovery complete to degree and order 15, i.e. a ratio of 34, compared 0.0059 and 0.0109 or a ratio of 1.8 for ll-SST observations (Table 2).

Figs. 5 and 6 display the condition numbers of the normal matrix and formal geoid error estimates for gravity field recoveries up to $l_{\max} = 50$ for the 15/1-repeat orbit, i.e. $l_{\max} > 3n_{\mathrm{r}} + 1$, and up to $l_{\max} = 65$ for the 31/2-repeat period, i.e. $l_{\max} > 2n_{\mathrm{r}} + 1$. The observable types include (1) geoid values, (2) radial orbit perturbations, (3) along-track orbit perturbations, (4) orbit perturbations in all directions (3D), (5) along-track diagonal gravity gradient component ($\mathit{\Gamma}_{\mathrm{xx}}$), (6) cross-track diagonal gravity gradient component ($\mathit{\Gamma}_{\mathrm{yy}}$), (7) radial diagonal gravity gradient component ($\mathit{\Gamma}_{\mathrm{zz}}$), and (8) all three diagonal gravity gradient components ($\mathit{\Gamma}_{\mathrm{xx+yy+zz}}$).

Table 2. Formal global geoid error (mm) (RMS, minimum and maximum) and the ratio of maximum and minimum geoid error at the equator (ρ_{eq}) for 15/1- and 31/2-repeat orbits. For $l_{\max} < n_{\mathrm{r}}/2$ the error is always constant as a function of longitude.

Obs.	$l_{\max}$	RMS	ρ_{eq}	min.	max.
15/1-repeat					
geoid	7	0.2632	1.00	0.1333	0.3200
geoid	8	0.3976	1.08	0.1419	0.9105
geoid	15	1.7342	12.54	0.1924	6.4213
ll-SST	7	0.0023	1.00	0.0013	0.0028
ll-SST	8	0.0027	1.00	0.0014	0.0034
ll-SST	15	0.0076	1.37	0.0059	0.0109
31/2-repeat					
geoid	15	0.3810	1.00	0.1350	0.4724
geoid	16	0.4092	1.00	0.1392	0.5314
geoid	32	1.4984	1.00	0.1922	4.9795
ll-SST	15	0.0033	1.00	0.0013	0.0038
ll-SST	16	0.0036	1.00	0.0014	0.0042
ll-SST	31	0.0108	1.00	0.0041	0.0139

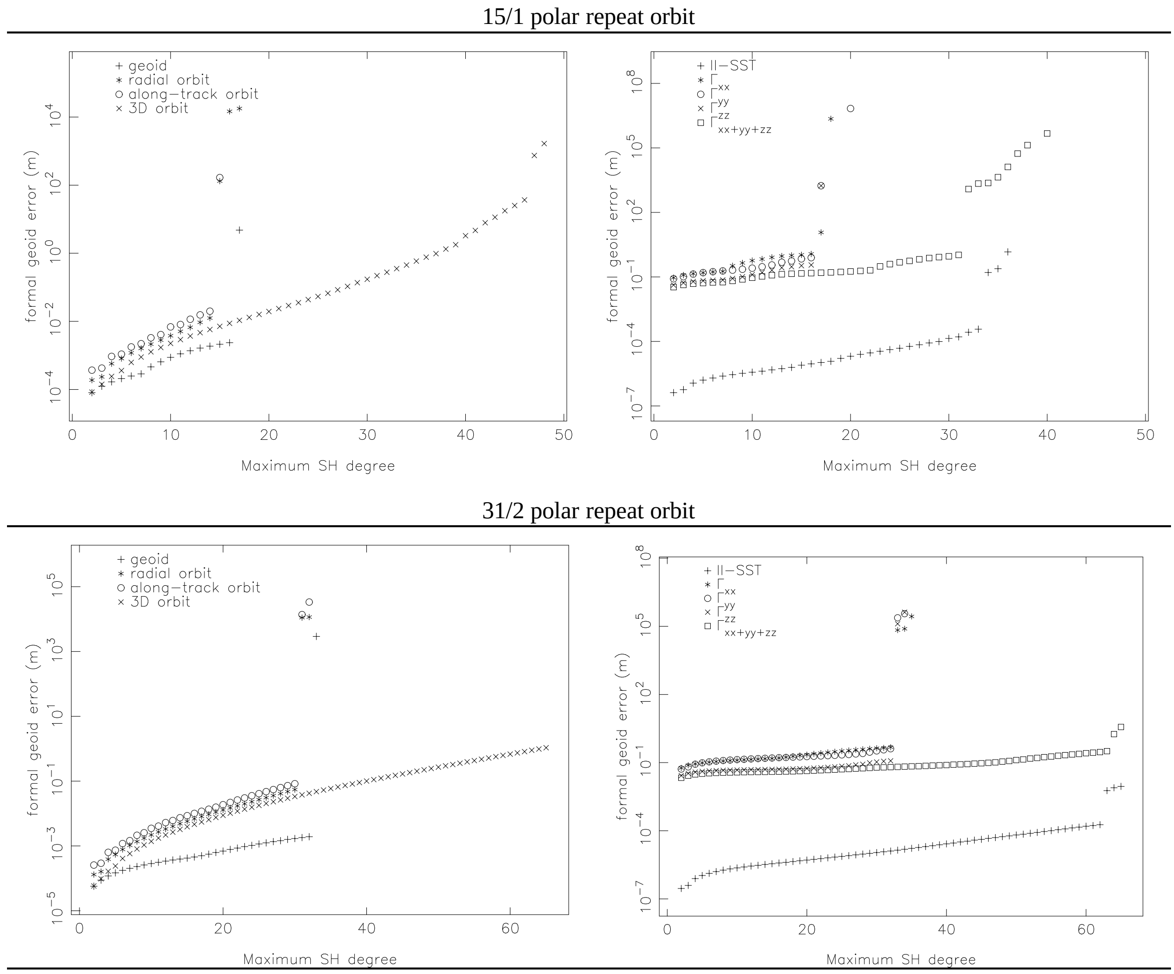

Fig. 6. Global RMS formal geoid error from the inverse of the normal equations as a function of the maximum retrieved spherical harmonic degree and the observation technique (the minimum degree is equal to 2).

It can be observed that for one-directional observables, such as geoid values, radial orbit perturbations, along-track perturbations, and one diagonal of the gravity gradients, the condition numbers display in general small jumps at $n_{\mathrm{r}}/2$ and large jumps at $n_{\mathrm{r}}+1$. The same can be observed for the associated geoid error estimate (provided the normal matrix was invertible). In other words, for such one-directional observables it seems like the maximum resolvable SH degree is equal to the number of revolutions $n_{\mathrm{r}}+1$ in a repeat period. When using ll-SST observations, combinations of orbit perturbations (3D) or combinations of SGG diagonal components, the normal matrix is stable up to at least $l_{\max}=2n_{\mathrm{r}}+1$. For the 3D combination of orbit perturbations, the condition number and associated formal geoid error estimate stays stable for $l_{\max}+1$ up to $3n_{\mathrm{r}}+1$, whereas for the combination of all three diagonal SGG components, this is still $2n_{\mathrm{r}}+1$.

Two questions that might now immediately be raised is why this is not $3n_{\mathrm{r}}+1$ for the combination of three SGG components as well and why it is $2n_{\mathrm{r}}+1$ for ll-SST observations, which is a one-directional observation type, namely along the line-of-sight between two trailing satellites. Concerning the SGG observations, it can be argued that the three diagonal components are not independent because the gravitational potential satisfies the Laplace equation, or $\mathit{\Gamma}_{\mathrm{xx}}+\mathit{\Gamma}_{\mathrm{yy}}+\mathit{\Gamma}_{\mathrm{zz}}=0$. Thus one diagonal SGG components can always be written as a linear combination of the other two. Thus, in fact only two independent components remain. Concerning the ll-SST observations, it can be argued that these observations are a modulated combination of along-track and radial orbit perturbations (Visser (2005)), assuming the two associated satellites fly in the same orbital plane.

5 Conclusions

Computations have shown that the Colombo-Nyquist rule in satellite geodesy, which predicts that the maximum resolvable degree is equal to half the number of orbital revolutions n_r in a repeat period of n_d nodal days, requires revision. Colombo's rule is correct in the sense that block-diagonal matrices are formed when $l_\mathrm{max} < n_\mathrm{r}/2$ and when organized per SH order, with no correlations between the orders. Colombo's rule is in general too pessimistic to infer statistical significance of SH coefficients in a gravity field model, i.e. solutions are possible where $l_\mathrm{max} \geq n_\mathrm{r}/2$ as is discussed in this paper. If the maximum degree of estimated SH coefficients is larger than $n_\mathrm{r}/2$, the gravity field solution will however no longer be homogeneous in the longitude direction for even parities of n_r and n_d. However, the Colombo-Nyqyist rule can be considered to be correct to some extent. Namely, as stated in the previous paragraph, the quality of recovered gravity field models is always homogeneous as a function of geographical longitude as long as $l_\mathrm{max} < n_\mathrm{r}/2$.

It was also found that the maximum resolvable degree does not depend on the parity of the number of revolutions and nodal days in a repeat orbit, but that the recovery error as a function of longitude does vary due to the increasing ground track density when traveling away from the equator. Finally, the maximum resolvable degree depends on the (combination of) observable type(s). In case of combinations of independent observables, this maximum degree can be one, two or three times the number of orbital revolutions in a repeat period (plus 1 if the minimum SH degree is taken equal to 2). Fortunately, in general gravity satellites carry a complement of observing instruments, including always GPS receivers in addition to for example ll-SST instruments or a gradiometer.

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